HADRONIC JOURNAL 49, 1 - 25 (2026)
DOI 10.29083/HJ.49.01.2026/SC1

THE QUANTUM THEORY OF THE PSEUDOSCALAR MESON

Simon Davis
Research Foundation of Southern California
8861 Villa La Jolla Drive #13595
La Jolla, CA 92039
sbdavis@resfdnsca.org

Received October 17, 2025
Revised January 14, 2026

Abstract

The quantum theory of a generalized theory of psuedoscalar mesons is studied. Renormalizability is demonstrated by cancelling the divergences in the momentum regulator Λ through renormalization of the coupling and the mass. The quantum corrections to the pseudoscalar mass are computed by adding the finite parts of the two-point fermion loop amplitudes. It is established that the quantum corrected mass m_q, equal to $m_{\eta'}$, can be generated from a classical mass $m = m_\eta$ for a specific value of the coupling.

Keywords: *semiclassical scales, generalized action, mass renormalization, eta prime meson, pseudoscalar meson coupling*

PACS: 12.38.Bx, 12.39.Fe, 12.39.Mk, 12.40.Yx

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1 Introduction

The common occurrence of couplings of pseudoscalar mesons to quarks is represented by terms such as $g\bar{\psi}i\gamma^5\phi\psi$. The action may be modified at semiclassical scales, however, as a result of the generalization of the equations for the spin-0 and spin-$\frac{1}{2}$ fields when the energy and momentum operators include another term for a real component of the canonical commutation relations. Since the new free field equations derived from the relativistic relation between energy and momentum are known under these conditions, the alteration to the kinetic terms can be given. The renormalizability of this theory will be studied with a momentum space propagator resulting from the modification of the field equations derived from the relativistic energy conservation rule.

It shall be demonstrated that there is an alternative physical interpretation of diagrams in the pseudoscalar theory. The integrals representing the diagrams are identical to those for the scalar Yukawa term except for the occurrence of the γ_5 matrix. It can be moved through the integrand until it has a similar form to the scalar theory. The only differences are a reversal of the sign of the momentum in the fermion propagator and a change of the chirality.

The superficial degree of divergence of the two-point pseudoscalar amplitude is 2 at all orders in perturbation theory. This infinity can be cancelled by a renormalization of the coupling g. The non-Lorentz invariant propagator and the difference between the limits of the energy and momentum integrals does not affect the power counting in the diagrammatic expansion of the scattering matrix. These features of the calculation do tend to introduce new functional dependences at lower orders in momentum cut-off Λ. There is a logarithmic subleading dependence on Λ in the

two-point amplitude, which can be absorbed into a renormalization of the pseudoscalar mass m_1. The self-energy graphs all have linear divergences that can be removed through a renormalization of the fermion mass m_2.

The finite quantum corrections to the pseudoscalar mass are evaluated by summing over the fermion loop diagrams with two pseudoscalar external lines. The number of diagrams of this type at n^{th} order in the perturbation series is known, and, when the amplitudes have magnitudes that are approximately equal, the correction can be computed. The mass m_1 can be set equal to the η meson mass. Then, the quantum corrected mass m_{1q} can be compared to the η' mass. An equality between m_{1q} and $m_{\eta'}$ is valid for a phenomenologically realistic value of the coupling in the pseudoscalar meson theory.

2 The Perturbative Diagrams of the Pseudoscalar Meson Theory

The Lagrangian of the Lorentz invariant pseudoscalar meson theory is

$$L = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) + \frac{1}{2}m_1^2\phi^2 + i\bar{\psi}(i\partial\!\!\!/ - m_2)\psi + ig\bar{\psi}\gamma_5\psi\phi. \tag{1}$$

At semiclassical scales, when a real component of the canonical commutation relations of the position and momenta may occur, a generalization of the Hermitian operators

$$\tilde{p}_i = \kappa_0\gamma^i\frac{\partial}{\partial x^2} - i\hbar\frac{\partial}{\partial x^i}$$
$$\tilde{E} = \kappa_0\gamma_5\frac{\partial}{\partial t} - i\hbar\frac{\partial}{\partial t}. \tag{2}$$

Tracing the relativistic formula for the energy, the wave equation of a free scalar field φ is

$$-3\frac{\partial^2\phi}{\partial t^2} - 5\sum_i \frac{\partial^2\varphi}{\partial x_i^2} + m^2\varphi = 0. \tag{3}$$

after setting $\kappa_0 = \hbar = 1$, and the momentum-space propagator is

$$i\Delta_F(E,\vec{p}) = \frac{15i}{5E^2 + 3|\vec{p}|^2 + 15m^2}. \tag{4}$$

Similarly, the new equation of the free spin-$\frac{1}{2}$ field is

$$(\sqrt{3}\gamma^0\partial_0 - \sqrt{5}i\gamma^i\partial_i - m)\psi = 0. \tag{5}$$

and the fermion propagator is generalized to

$$iS_{\alpha\beta}(E.\vec{p}) = \frac{5i\left(-\kappa_0\gamma_5 + i\hbar\mathbb{I}\right)(\gamma\cdot p + m))_{\alpha\beta}}{5E^2 + 3|\vec{p}|^2 + 14m^2}. \tag{6}$$

The pseudoscalar meson action then would be

(7)

$$S_{semiclassical} = \int d^4x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{5}{6}\sum_i\left(\frac{\partial\phi}{\partial x_i}\right)^2 + \frac{1}{6}m_1^2\phi^2 + \sqrt{3}\bar{\psi}\left(\gamma^0\partial_0 - \sqrt{5}i\gamma^i\partial_i - m_2\right)\psi + ig\bar{\psi}\gamma_5\psi\phi\right].$$

Given that there is an interaction term $g\bar{\psi}i\gamma_5\psi\phi$, the diagram at first order in perturbation theory with two vertices and one fermion loop may be evaluated. There are two such amplitudes, related by reflecting the internal fermion lines about the central axis of the diagram. It is necessary to multiply by this factor of 2 and -1, required for a fermion loop, to evaluate the entire two-point pseudoscalar amplitude at this order

(8)

$$I_1(p) =$$

$$-2g^2 \int \frac{d^4k}{(2\pi)^4} Tr\left[\frac{5i(-\kappa_0\gamma_5 + i\hbar\mathbb{I})(\gamma\cdot k + m_2)i\gamma_5}{5E_k^2 + 3|\vec{k}|^2 + 15m_2^2} \frac{5i(-\kappa_0\gamma_5 + i\hbar\mathbb{I})(\gamma\cdot(p-k) + m_2)i\gamma_5}{5E_{p-k}^2 + 3|\vec{p}-\vec{k}|^2 + 15m_2^2}\right]$$

$$\delta(p_i - p_f)$$

$$= -50g^2 \int \frac{d^4k}{(2\pi)^4} Tr\left[\frac{(-\kappa_0\gamma_5 + i\hbar\mathbb{I})(\gamma\cdot k + m)\gamma_5}{5E_k^2 + 3|\vec{k}|^2 + 15m_2^2} \frac{(-\kappa_0\gamma_5 + i\hbar\mathbb{I})(\gamma\cdot(p-k) + m)\gamma_5}{5E_{p-k}^2 + 3|\vec{p}-\vec{k}|^2 + 15m_2^2}\right]$$

$$\delta(p_i - p_f).$$

The number of powers in the numerator is four and the number in the denominator is two. Therefore, the amplitude is quadratically divergent. Since the momentum space propagators have the same dimension in this field theory, the renormalization procedure would be similar.

The γ_5 factors occur four times in the integral. The second γ_5 matrix from the interaction term can be moved to the beginning of the product until it multiplies the $(-\kappa_0\gamma_5 + i\hbar\mathbb{I})$ matrix by a cyclic permutation. The momentum will be reversed in the propagator, while the product of the matrices is $(-\kappa_0\gamma_5 + i\hbar\mathbb{I})i\gamma_5 = -i\kappa_0 I - \hbar\gamma_5$ which equals $-i\mathbb{I} - \gamma_5$ in units with κ_0 and $\hbar$ equated to one. It follows that, if κ_0 is set equal to $\hbar$ and the fermions have one chirality in the pseudoscalar theory, the diagrams could be evaluated identically to that of the scalar model with the fermions having the opposite chirality. Then

(9)
$$Tr\ [(-\gamma_5 + i\mathbb{I})\,(\gamma\cdot k + m_2)\gamma_5\,(-\gamma_5 + i\mathbb{I})\,(\gamma\cdot(p-k) + m_2)\gamma_5]$$
$$= Tr\ [(-\mathbb{I} + i\gamma_5)\,(\gamma\cdot k + m_2)\,(-\mathbb{I} + i\gamma_5)\,(\gamma\cdot(p-k) + m_2)]$$
$$= 2\ Tr\ ((\gamma\cdot k)(\gamma\cdot(p-k))) = 8k\cdot(p-k)$$

and the amplitude is

(10)

$$-\frac{5^2g^2}{2^2\pi^4}\int dE_k|\vec{k}|^2d|\vec{k}|\int_0^{2\pi}d\phi$$
$$\int_0^{\pi}d\theta\ sin\ \theta\frac{4(E_kE_{p-k}-|\vec{k}||\vec{p}|cos\ \theta+|\vec{k}|^2)}{(5E_k^2+3|\vec{k}|^2+15m_2^2)(5E_{p-k}^2+3(|\vec{k}|^2+|\vec{p}|^2-2|\vec{k}||\vec{p}|cos\ \theta))}$$
$$=-\frac{5^2g^2}{2\pi^3}\int dE_k|\vec{k}|^2d|\vec{k}|$$
$$\int_{-1}^{1}d(cos\ \theta)\frac{4(E_kE_{p-k}+|\vec{k}|^2)-4|\vec{k}||\vec{p}|cos\ \theta}{(5E_k^2+3|\vec{k}|^2+15m_2^2)[5E_{p-k}^2+3(|\vec{k}|^2+|\vec{p}|^2)+15m_2^2-6|\vec{k}||\vec{p}|cos\ \theta]}$$

Since

(11)
$$\int dx\ \frac{a_1+b_1x}{a_2+b_2x}=\frac{b_1}{b_2}x+\frac{1}{b_2}\left(a_1-\frac{b_1}{b_2}a_2\right)ln(a_2+b_2x),$$

(12)

$$\int_{-1}^{1}d(cos\ \theta)\frac{[4(E_kE_{p-k}+|\vec{k}|^2)]-4|\vec{k}||\vec{p}|cos\ \theta}{(5E_k^2+3|\vec{k}|^2+15m_2^2)[5E_{p-k}^2+3(|\vec{k}|^2+|\vec{p}|^2+15m_2^2-6|\vec{k}||\vec{p}|cos\ \theta]}$$
$$=\frac{1}{5E_k^2+3|\vec{k}|^2+15m_2^2}\left[\frac{4}{3}-\frac{1}{6|\vec{k}||\vec{p}|}\left(4(E_kE_{p-k}+|\vec{k}|^2)\right.\right.$$
$$\left.\left.-\frac{2}{3}(5E_{p-k}^2+3(|\vec{k}|^2+|\vec{p}|^2)+15m_2^2)\right)\ ln\left(\frac{5E_{p-k}^2+3(|\vec{k}|+|\vec{p}|)^2+15m_2^2}{5E_{p-k}^2+3(|\vec{k}|-|\vec{p}|)^2+15m_2^2}\right)\right].$$

The integral of the first term is

$$-\frac{2\cdot5^2g^2}{3^2\pi^3}\int_0^{\sqrt{\Lambda^2+m_2^2}}dE_k\int_0^{\Lambda}d|\vec{k}|\left(1-\frac{\frac{5}{3}E_k^2+5m_2^2}{\frac{5}{3}E_k^2+|\vec{k}|^2+5m_2^2}\right)$$
$$=-\frac{2\cdot5^2g^2}{3^2\pi^3}\left[\Lambda\sqrt{\Lambda^2+m_2^2}-\int_0^{\sqrt{\Lambda^2+m_2^2}}dE_k\sqrt{\frac{5}{3}E_k^2+5m_2^2}\ arc\ tan\ \frac{\Lambda}{\sqrt{\frac{5}{3}E_k^2+5m_2^2}}\right].$$

The range $[0, \sqrt{\Lambda^2+m^2}]$ can be divided into $\left[0, \sqrt{\frac{3}{5}\Lambda^2-3m^2}\right]$ and $\left[\sqrt{\frac{3}{5}\Lambda^2-3m^2}, \sqrt{\Lambda^2+m^2}\right]$ such that

(13)

$$\frac{2\cdot 5^2 g^2}{3^2\pi^3}\int_0^{\sqrt{\Lambda^2+m^2}} dE_k\sqrt{\frac{5}{3}E_k^2+5m^2}\; arc\,tan\left(\frac{\Lambda}{\sqrt{\frac{5}{3}E_k^2+5m^2}}\right)$$

$$=\frac{2\cdot 5^2 g^2}{3^2\pi^3}\int_0^{\sqrt{\frac{3}{5}\Lambda^2-3m^2}} dE_k\sqrt{\frac{5}{3}E_k^2+5m^2}\left[\frac{\pi}{2}-\frac{\sqrt{\frac{5}{3}E_k^2+5m^2}}{\Lambda}+\frac{\left(\frac{5}{3}E_k^2+5m^2\right)^{\frac{3}{2}}}{3\Lambda^3}+\ldots\right]$$

$$+\frac{2\cdot 5^2 g^2}{3^2\pi^3}\int_{\sqrt{\frac{3}{5}\Lambda^2-3m^2}}^{\sqrt{\Lambda^2+m^2}} dE_k\sqrt{\frac{5}{3}E_k^2+5m^2}\left[\frac{\Lambda}{\sqrt{\frac{5}{3}E_k^2+5m^2}}-\frac{1}{3}\frac{\Lambda^3}{\left(\frac{5}{3}E_k^2+5m^2\right)^{\frac{3}{2}}}+\ldots\right]$$

$$=\frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[\frac{\pi}{2}\sqrt{\frac{5}{3}}\frac{\left(\frac{3}{5}\Lambda^2-3m^2\right)}{2}-\frac{5}{9}\frac{\left(\frac{3}{5}\Lambda^2-3m^2\right)^{\frac{3}{2}}}{\Lambda}+\frac{5}{27}\frac{\left(\frac{3}{5}\Lambda^2-3m^2\right)^{\frac{5}{2}}}{\Lambda^3}-\ldots.\right]\right.$$

$$\left.+\left[\Lambda^2+m^2-\left(\frac{3}{5}\Lambda^2-3m^2\right)-\frac{1}{5}\Lambda^3\left(\frac{1}{\sqrt{\frac{3}{5}\Lambda^2-3m^2}}-\frac{1}{\sqrt{\Lambda^2+m^2}}\right)+\ldots\right]\right\}+O(ln\;\Lambda)$$

$$=\frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[\frac{\pi}{4}\sqrt{\frac{3}{5}}-\frac{1}{5}\sqrt{\frac{3}{5}}+\frac{1}{15}\sqrt{\frac{3}{5}}-\ldots\right]-\left[\frac{2}{5}-\frac{1}{5}\left(\sqrt{\frac{5}{3}}-1\right)+\ldots\right]\right\}\Lambda^2+O(ln\;\Lambda)$$

In the limit $p^\mu \to 0$,

$$I_1(0)=-\frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[1-\left[\frac{\pi}{4}\sqrt{\frac{3}{5}}-\frac{1}{5}\sqrt{\frac{3}{5}}+\frac{1}{15}\sqrt{\frac{3}{5}}-\ldots\right]+\left[\frac{2}{5}-\frac{1}{5}\left(\sqrt{\frac{5}{3}}-1\right)+\ldots\right]\right]\Lambda^2\right.$$

$$\left.+O(ln\;\Lambda)\right.$$

$$lim_{p\to 0}\frac{1}{8|\vec{p}|}\int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k\int_0^{\Lambda} d|\vec{k}|\;\frac{|\vec{k}|}{5E_k^2+3|\vec{k}|^2+15m^2}$$

$$\left[\left(4E_kE_{p-k}+2|\vec{k}|^2-\frac{10}{3}E_{p-k}^2-2|\vec{p}|^2-10m_2^2\right)\right.$$

$$\left.\left.ln\left(\frac{5E_{p-k}^2+3(|\vec{k}|+|\vec{p}|)^2+15m_2^2}{5E_{p-k}^2+3(|\vec{k}|-|\vec{p}|)^2+15m_2^2}\right)\right]\right\}$$

$$= \frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[1-\left[\frac{\pi}{4}\sqrt{\frac{3}{5}}-\frac{1}{5}\sqrt{\frac{3}{5}}+\frac{1}{15}\sqrt{\frac{3}{5}}-...\right]+\left[\frac{2}{5}-\frac{1}{5}\left(\sqrt{\frac{5}{3}}-1\right)+...\right]\right]\Lambda^2\right.$$
$$\left.+\frac{3}{2}\int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k \int_0^{\Lambda} d|\vec{k}|\frac{|\vec{k}|^2}{(5E_k^2+3|\vec{k}|^2+15m_2^2)^2}\left(-\frac{22}{3}E_k^2+2|\vec{k}|^2-10m_2^2\right)\right\}$$
$$+O(ln\ \Lambda).$$

Since

$$\int_0^{\Lambda} d|\vec{k}|\frac{|\vec{k}|^2}{(5E_k^2+3|\vec{k}|^2+12m_2^2)^2}\left(-\frac{22}{3}E_k^2+2|\vec{k}|^2-10m_2^2\right)$$
$$=\frac{1}{9}\left(-\frac{22}{3}E_k^2-10m_2^2\right)\int_0^{\Lambda} d|\vec{k}|\ \frac{1}{|\vec{k}|^2}\frac{1}{\left(1+\frac{\frac{5}{3}E_k^2+5m_2^2}{|\vec{k}|^2}\right)^2}$$
$$+\frac{2}{9}\int_0^{\Lambda} d|\vec{k}|\frac{|\vec{k}|^4}{\left(\frac{5}{3}E_k^2+|\vec{k}|^2+5m_2^2\right)^2}$$
$$=\frac{2}{9}\Lambda+\left(\frac{14}{9}E_k^2+\frac{30}{9}m_2^2\right)1\Lambda+O\left(\frac{1}{\Lambda^3}\right),$$

(14)

$$I_1(0)=-\frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[1-\left[\frac{\pi}{4}\sqrt{\frac{3}{5}}-\frac{1}{5}\sqrt{\frac{3}{5}}+\frac{1}{15}\sqrt{\frac{3}{5}}-...\right]+\left[\frac{2}{5}-\frac{1}{5}\left(\sqrt{\frac{5}{3}}-1\right)+...\right]\right.\right.$$
$$\left.\left.+\frac{3}{2}\left[\left[\frac{2}{9}+\frac{14}{27}+...\right]\right\}\Lambda^2+O(ln\ \Lambda)+O(1).\right.\right.$$

Another conventional method for evaluating integrals of this type may be considered. Given the momentum variable $\hat{p}=(\hat{E},\vec{p})=\left(\frac{E}{\sqrt{3}i},\frac{\vec{p}}{\sqrt{5}}\right)$,

$$(15)\qquad \frac{1}{(5E_{p-k}^2+3|\vec{p}-\vec{k}|^2+15m_2^2)(5E_k^2+3|\vec{k}|^2+15m_2^2)}$$
$$=\frac{225}{\left(\left(\frac{E_{p-k}}{\sqrt{3}i}\right)^2-\left(\frac{|\vec{p}-\vec{k}|}{\sqrt{5}}\right)^2+m_2^2\right)\left(\left(\frac{E_k}{\sqrt{3}i}\right)^2-\left(\frac{|\vec{k}|}{\sqrt{5}}\right)^2+m_2^2\right)}$$
$$=\frac{225}{((\hat{p}-\hat{k})^2-m_2^2)(\hat{k}^2-m_2^2)}$$

Let $q=\hat{k}-x\hat{p}$, $D=x(1-x)\hat{p}^2$ and $\tilde{x}=x\hat{p}(-2\hat{k}+\hat{p})+\hat{k}^2-m_2^2$. Then

$$(16)\qquad \int_0^1 dx\frac{1}{(y^2+D)^2}=\int_0^1\frac{dx}{[x\hat{p}(-2\hat{k}+\hat{p})+\hat{k}^2-m_2^2]^2}$$
$$=\frac{\int_{\hat{k}^2-m_2^2}^{(\hat{p}-\hat{k})^2-m_2^2}\frac{d\tilde{x}/(\hat{p}(-2\hat{k}+\hat{p}))}{\tilde{x}^2}}{[x\hat{p}(-2\hat{k}+\hat{p})+\hat{k}^2-m_2^2]^2}$$
$$=\frac{1}{\hat{p}(-2\hat{k}+\hat{p})}\int_{\hat{k}^2-m_2^2}^{(\hat{p}-\hat{k})^2-m^2}\frac{d\tilde{x}}{\tilde{x}^2}$$
$$=\frac{1}{\hat{p}(-2\hat{k}+\hat{p})}\left[\frac{1}{\hat{k}^2-m_2^2}-\frac{1}{(\hat{p}-\hat{k})^2-m_2^2}\right]$$
$$=\frac{1}{((\hat{p}-\hat{k})^2-m_2^2)(\hat{k}^2-m_2^2)}.$$

It follows that the amplitude is

$$(17)\qquad -2^2 3^2 5^4 g^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\delta(p_i-p_f)\frac{5k\cdot(p-k)-m_2^2}{(q^2+D)^2}$$

Since $(E,\vec{p})=(\sqrt{3}i\hat{E},\sqrt{5}\hat{p})$,

$$(18)\qquad 5k\cdot(p-k)-3m^2=-[15E_{\hat{k}}(E_{\hat{p}-\hat{k}})+25\vec{\hat{k}}\cdot(\vec{\hat{p}}-\vec{\hat{k}})+3m_2^2],$$

and the integral is

$$(19)\qquad 2^2 3^2 5^4 g^2\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\frac{15E_{\hat{k}}(E_{\hat{p}-\hat{k}})+25\vec{\hat{k}}\cdot(\vec{\hat{p}}-\vec{\hat{k}})+3m_2^2}{(q^2+D)^2}.$$

The expression in the numerator also has some dependence on q. Therefore, this technique does not allow an immediate evaluation of the integral. Instead, it is necessary to represent the numerator in terms of partial derivatives of an exponential function of the momenta.

The superficial degree of divergence of all of the two-point pseudoscalar diagrams is 2. At $O(g^4)$, there are three diagrams represented by the insertion of an internal spin-0 line, and a pseudoscalar line attached to the exterior of the fermion loop. Let the amplitudes for the first graph and the other two diagrams be $I_2^{(1)}(p)$ and $I_2^{(2)}(p)$ respectively. Then, the renormalized coupling in the limit of vanishing external momentum is

$$g(\Lambda) = g + A_1\Lambda^2 + A_2\Lambda^2 + ... \tag{20}$$

where $A_1 = lim_{\Lambda\to\infty} - \frac{I_1(0)}{\Lambda^2}$, $A_2 = lim_{\Lambda\to\infty} \frac{-\frac{1}{2}}{(} I_2^{(1)}(0) + 2I_2^{(2)}(0))\Lambda^2$, ... after including the inverse factorial factor in the expansion of the scattering matrix. The number of diagrams at each order is estimated from the inclusion of series of spin-0 lines and fermion loops in these graphs or sets of of a minimum of two pseudoscalar propagators connecting different spin-$\frac{1}{2}$ loops. Consequently, A_n is bounded and the renormalization of the coupling is well-defined with counterterms cancelling in the quadratic divergences in the amplitudes. The logarithmic divegences may be absorbed into a renormalization of the pseudosclar mass m_1.

The fermion self-energy diagram has two vertices one pseudoscalar propagator and one fermion propagator. The integral is

(21)

$$i\Sigma_{f1} = -g^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}_r(\vec{p}) \frac{\left[3 \cdot 5^2 i\gamma_5 \left(-\gamma_5 + i\mathbb{I}\right)(\gamma \cdot (p-k) + m_2) i\gamma_5\right]}{(5E^2_{p-k} + 3|\vec{p}-\vec{k}|^2 + m_2^2)(5E_k^2 + 3|\vec{k}|^2 + m_1^2)} u_r(\vec{p})$$

$$= 3 \cdot 5^2 \; i \; g^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}_r(\vec{p}) \frac{\left[\left(\mathbb{I} + \frac{i}{2}\gamma_5\right)(\gamma \cdot (p-k) - m_2)\right]}{(5E^2_{p-k} + 3|\vec{p}-\vec{k}|^2 + m_2^2)(5E_k^2 + 3|\vec{k}|^2 + m_1^2)} u_r(\vec{p}).$$

Since $(\gamma \cdot p - m)u_r(p) = 0$,

(22)

$$i\Sigma_{f1} = -\frac{25}{6} ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_r(p)(\mathbb{I} + i\gamma_5)\gamma \cdot k \; u_r(p)}{(5E_k^2 + 3|\vec{k}|^2 + 15m_1^2)(5E^2_{p-k} + 3|\vec{p}-\vec{k}|^2 + 15m + 2^2)}$$

which is linearly divergent. Amongst the one-particle irreducible amplitudes there will be rainbow diagrams and fermion loops in the interior of this self-energy graph. An n-loop rainbow diagram has a superficial degree of divergence equal to $4n - 2n - (2n - 1) = 1$. This infinity may be absorbed into a fermion mass renormalization

(23) $$m_2(\Lambda) = m_2 + (B_1 + B_2 + ...)\Lambda$$

with $B_1 = \Sigma_{f1}(0)$, $B_2 = \frac{1}{21}\Sigma_{f2}(0)$, ...

3 The Mass of the Pseudoscalar Meson

The mass of the pseudoscalar meson would received contributions from the two-point diagrams at higher orders in perturbation theory. After the cancellation of the quadratic infinities, there remain finite terms which can be added to the momentum-space propagator. Even though the renormalizability analysis does not

required one-particle reducible diagrams, these graphs are still included in the evaluation of the scattering amplitudes. Given the 0^- spin coupling formula and gluon condensate contribution, yielding a mass $m_0 + m_{gc}$ [2], the sum of quark-anti-quark annihilation diagrams results in a sum in quadrature of $m_0 + m_{gc}$ and the octet mass [3] for $m_{\eta'}$. An evaluation of only a subset of the the diagrams of the pseudoscalar meson theory at higher orders in perturbation results in the quadrature formula for the shift in the massive propagator even though it yields a precise value of the mass of the η' meson. There are 2^n different diagrams with n quark-anti-quark loops, each having a symmetry factor of 2^{n-1}. The coefficient of the amplitude at this order then would be $\frac{2^{2n-1}}{n!}$. Within the context of a conventional quantum field theoretic computation,

(24)

$$\frac{1}{p^2-(m_0+m_{gc})^2} \rightarrow \frac{1}{p^2-(m_0+m_{gc})^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{1}{4}\frac{4m_8^2}{(p^2-(m_0+m_{gc})^2}\right)^n\right]$$
$$= \frac{1}{p^2-(m_0+m_{gc})^2}\frac{1}{1-\frac{m_8^2}{p^2-((m_0+m_{gc})^2+m_8^2)}} = \frac{1}{p^2-((m_0+m_{gc})^2+m_8^2}.$$

The factor of 4 multiplying m_8^2 arises from trace of the identity matrix. It remains to be established if the effect of the gluon condensate and octet mass can be theoretically explained by the pseudoscalar meson theory.

The pseudoscalar propagator is modified to

(25)

$$\Delta_F(p)_{ren} = \frac{15i}{5E_p^2+3|\vec{p}|^2+15m_1^2}\left[1+\widehat{I_1(p)}\right.$$
$$\left.+\left(\widehat{I_2^{(1)}(p)}+2\widehat{I_2^{(2)}(p)}+\widehat{I_2^{(3)}(p)}\right)+...+...\right]\Big|_{finite}$$

where the normalized amplitude $\widehat{I_1(p)}$ is dimensionless, and therefore, the amplitudes expression must be divided by μ_{QCD}^2, where $\mu_{QCD} \approx 200 MeV/c^2$. and $I_2^{(3)}(p)$ is the one-particle reducible amplitude with two fermion loops in sequence, and

The limit, as the external momentum vanishes, equals

(26)

$$lim_{p\to 0}\Delta_F(0)_{ren} = \frac{3i}{4m_1^2}\left[1+\left(\widehat{I_1(0)}+\left(\widehat{I_2^{(1)}(0)}+2\widehat{I_2^{(2)}(0)}+\widehat{I_2^{(3)}(0)}\right)+...\right)\right]\Bigg|_{finite}$$

The factors multiplying the amplitudes will be given by the product of $\frac{1}{n!}$, and the number of the interchanges of the lines that leave Invariant the one-particle irreducible subgraphs of a diagram. Therefore, the n-loop one-particle, reducible diagram consisting of a sequence of fermion loops, is multiplied by $\frac{2^n}{n!}$, and a set of reducible amplitudes is given by

$$(27)\quad I_{nseq.\ red.}(p)\Big|_{finite} \equiv (-1)^{n+1}\frac{2^{n-1}}{n!}I_1(p)^n_{finite}\left(\frac{15i}{5E_p^2+3|\vec{p}|^2+15m_1^2}\right)^{n-1}$$
$$= -\frac{2^{n-1}}{n!}|I_1(p)_{finite}|^n\left(\frac{15i}{5E_p^2+3|\vec{p}|^2+15m_1^2}\right)^{n-1}$$

Then

$$(28)\quad \sum_{n=1}^{\infty} I_{nseq.\ red.}(p)\Big|_{finite} = \sum_{n=1}\frac{(-2\cdot 15i)^{n-1}}{n!}\frac{I_1(p)^n_{finite}}{(5E_p^2+|\vec{p}|^2+15m_1^2)^{n-1}}$$
$$= \frac{5E_p^2+|\vec{p}|^2+15m_1^2}{2\cdot 15}i\left[e^{\frac{-2\cdot 15iI_1(p)_{finite}}{5E_p^2+3|\vec{p}|^2+15m_1^2}}-1\right]$$

when tends to

$$(29)\quad \frac{2}{3}m_1^2 i\left[e^{\frac{-3iI_1(p)_{finite}}{2m_1^2}}-1\right]$$

as $|\vec{p}| \to 0$ and

(30)

$$\Delta_F(0)_{ren} = \frac{3i}{4m_1^2}\left[1 + \frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2} i\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right)\right.$$
$$\left.\left(\frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2}\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right) + (\widehat{I_2^{(1)}(0)} + 2\widehat{I_2^{(2)}(0)})_{finite} + \cdots\right)\right)\Bigg].$$

(31)

since one-particle irreducible subgraphs can be included within separate fermion loops in a a one-particle reducible amplitude. Then

(32)

$$Im\ \Delta_F(0)_{ren.} = \frac{3}{4m_1^2}\left[1 + Im\left[\frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2}\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right)\left(\frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2}\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right)\right.\right.\right.$$
$$\left.\left.\left. + \left(\widehat{I_2^{(1)}(0)} + 2\widehat{I_2^{(2)}(0)}\right)_{finite} + \cdots\right)\right]\right]$$

and

(33)

$$m_{1q}^2 = m_1^2\left[1 + Im\left[\frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2}\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right)\right.\right.$$
$$\left.\left.\left(\frac{2}{3}\frac{m_1^2}{\mu_{QCD}^2}\left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1\right) + \left(\widehat{I_2^{(1)}(0)} + 2\widehat{I_2^{(2)}(0)}\right)_{finite} + \cdots\right)\right)\right]\Bigg]^{-1}.$$

Since the number of vertices in each pseudoscalar two-point amplitude is even, the trace of the gamma matrices is real in a fermion loop and the pseudoscalar propagator is imaginary, $I_n^{(m_n)}(0)$ is real when there is an even number of internal pseudocalar propagators and imaginary if an odd number is present. The interior of the fermion loop is partitioned into n regions if there are $n-1$ pseudocalar internal

lines. These diagrams constitute a single set of fermion loop diagrams. When an internal pseudoscalar lines generate another fermion loop, two pseudoscalar propagators are introduced, yielding another imaginary unit. Then $I_n(0) = \sum_{m_n} I_n^{(m_n)}(0)$ is real only for odd n and

(34)

$$m_{1q} = m_1 \left[1 + Im \left[\frac{2}{3} \frac{m_1^2}{\mu_{QCD}^2} \left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1 \right) \left(\frac{2}{3} \frac{m_1^2}{\mu_{QCD}^2} \left(e^{\frac{-3iI_1(0)_{finite}}{2m_1^2}} - 1 \right) + \sum_{\substack{n \geq 3 \\ n \ odd}} \widehat{I_n(0)}_{non\ seq.\ red.,\ finite} + \cdots \right) \right] \right]^{-\frac{1}{2}}$$

where $I_n(p)_{non\ seq.\ red.}$ represents amplitudes of all n-loop diagrams excluding those consisting of a sequence of fermion loops with no other internal or attached lines.

There are nearly 2^n different types of one-particle irreducible two-point pseudoscalar diagrams at n^{th} order in perturbation theory. The overall sign depends on the the number of interior fermion loops. The partitioning of the n loops by $n-1$ gluon lines can be subdivided further into $n_k - 1$ lines in the k^{th} loop, with the k^{th} loop being connected to two pseudoscalar lines. When there are one outermost and $\ell - 1$ fermion loops, $\sum_{k=2}^{\ell}(n_k - 1) + 2(\ell - 1) = \sum_{k=1}^{\ell} n_k + \ell - 2 = n - 1$ or $\sum_{k=2}^{\ell} n_k = n - \ell - 1$, and the overall sign is $(-1)^\ell$. The maximum value of ℓ is $1 + \frac{n-1}{2}$ for odd n, and there are $\binom{n}{\ell}$ types of diagrams with ℓ fermion loops. Given an approximate equality of the magnitudes of the higher-order amplitudes, the contribution of the different types receives a factor of $\sum_{\ell=1}^{\frac{n+1}{2}} (-1)^\ell \binom{n-1}{\ell-1} = 2^{\frac{n-1}{2}} cos\left(\frac{(n-1)\pi}{4}\right)$. This estimate is based on the validity of the field theory limit of the eikonal approximation of the string scattering matrix in the evaluation of

the finite part of the amplitude [4][5]. The coefficient of the one-particle irreducible n-loop amplitude, required for equality with $\sum_{m_n} I_n^{m_n}(0)_{1PI}$, then tends to $\frac{2^{n-1}}{n!} \times 2^{\frac{n-1}{2}} cos\left(\frac{(n-1)\pi}{4}\right) = \frac{2^{\frac{3(n-1)}{2}}}{n!} cos\left(\frac{(n-1)\pi}{4}\right)$. Consequently,

$$(35) \quad \sum_{\substack{n\geq 3 \\ n\ odd}} \widehat{I_n(0)}_{non\ seq.\ red.,finite} \approx - \sum_{\substack{n\geq 3 \\ n\ odd}} \frac{2^{\frac{3(n-1)}{2}}}{n!} cos\left(\frac{(n-1)\pi}{4}\right) \frac{|\widehat{I_1(0)}_{finite}|^n}{\mu_{QCD}^2} \cdot \left(\frac{3i}{4m_1^2}\right)^{n-1}$$

$$= -\frac{|I_1(0)_{finite}|}{\mu_{QCD}^2} \sum_{s=1}^{\infty} \frac{1}{(4s+1)!} \left(\frac{3}{\sqrt{2}} \frac{|I_1(0)_{finite}|}{m_1^2}\right)^{4s}$$

$$= -\frac{|I_1(0)_{finite}|}{\mu_{QCD}^2} \left[-1 + \frac{m_1^2}{3\sqrt{2}|I_1(0)_{finite}|} \left(sin\left(\frac{3}{\sqrt{2}} \frac{|I_1(0)_{finite}|}{m_1^2}\right) + sinh\left(\frac{3}{\sqrt{2}} \frac{|I_1(0)_{finite}|}{m_1^2}\right)\right)\right]$$

Therefore,

(36)

$$m_{1q} \approx m_1 \left\{ 1 - \frac{2}{3} \frac{m_1^2}{\mu_{QCD}^2} sin\left(\frac{3|I_1(0)_{finite}|}{2m_1^2}\right) \left[\frac{2}{3} \frac{m_1^2}{\mu_{QCD}^2} sin\left(\frac{3|I_1(0)_{finite}|}{2m_1^2}\right) + \frac{|I_1(0)_{finite}|}{\mu_{QCD}^2} \left[1 - \frac{m_1^2}{3\sqrt{2}|I_1(0)_{finite}|} \left(sin\left(\frac{3}{\sqrt{2}} \frac{|I_1(0)_{finite}|}{m_1^2}\right) + sinh\left(\frac{3}{\sqrt{2}} \frac{|I_1(0)_{finite}}{m_1^2}\right)\right)\right]\right]\right\}^{-\frac{1}{2}}.$$

The $O(1)$ term in Eq.(10) is given by the expansion of the radicals in Eq.(14)

$$(37) \quad \sqrt{\frac{5}{3}E_k^2 + 5m_2^2} = \sqrt{\frac{5}{3}E_k\left[1 + \frac{1}{6}\frac{m_2^2}{E_k^2} + ...\right]} = \sqrt{\frac{5}{3}}E_k + \frac{1}{6}\sqrt{\frac{5}{3}}\frac{m_2^2}{E_k} - ...$$

$$\sqrt{\frac{3}{5}\Lambda^2 - 3m_2^2} = \sqrt{\frac{3}{5}}\Lambda - \frac{\sqrt{15}}{2}\frac{m_2^2}{\Lambda} + ...$$

and the fractions in Eq.(11)

$$(38) \qquad \frac{|\vec{k}|^2}{(5E_k^2+3|\vec{k}|^2+15m_2^2)^2} = \frac{1}{9|\vec{k}|^2} - \frac{2}{9}\frac{\frac{5}{3}E_k^2+5m_2^2}{|\vec{k}|^4} + \ldots$$
$$\frac{|\vec{k}|^4}{(5E_k^2+3|\vec{k}|^2+15m_2^2)^2} = \frac{1}{9} - \frac{2}{9}\frac{\frac{5}{3}E_k^2+5m_2^2}{|\vec{k}|^2} + \ldots .$$

together with

(39)

$$\int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k \left(-\frac{22}{3}E_k^2 - 10m_2^2\right) \int_0^{\Lambda} d|\vec{k}| \left[\frac{1}{9|\vec{k}|^2} - \frac{2}{9}\frac{\left(\frac{5}{3}E_k^2+5m_2^2\right)}{|\vec{k}|^4} + \ldots\right]$$
$$+ \frac{2}{9}\int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k \int_0^{\Lambda} d|\vec{k}| \left[1 - 2\frac{\left(\frac{5}{3}E_k^2+5m_2^2\right)}{|\vec{k}|^2} + \ldots\right]$$
$$= \int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k \left(-\frac{22}{3}E_k^2 - 10m_2^2\right) \left[-\frac{1}{9\Lambda} + \frac{2}{27\Lambda^3}\left(\frac{5}{3}E_k^2+5m_2^2\right) + \ldots\right]$$
$$+ \frac{2}{9}\int_0^{\sqrt{\Lambda^2+m_2^2}} dE_k \left[\Lambda + 2\left(\frac{5}{3}E_k^2+5m_2^2\right)\frac{1}{\Lambda} - \ldots\right].$$
$$= -\frac{1}{9\Lambda}\left[-\frac{22}{3}\left(\Lambda^2+m_2^2\right)^{\frac{3}{2}} - 10m_2^2\left(\Lambda^2+m_2^2\right)^{\frac{1}{2}}\right] + \frac{2}{27\Lambda^3}\left[\frac{2\cdot 5\cdot 11}{9}\left(\Lambda^2+m_2^2\right)^{\frac{5}{2}}\right.$$
$$\left. - \frac{160}{9}m_2^2\left(\Lambda^2+m_2^2\right)^{\frac{3}{2}} - \ldots.\right]$$
$$+ \frac{2}{9}\Lambda(\Lambda^2+m_2^2)^{\frac{1}{2}} + \frac{4}{9\Lambda}\left(\frac{5}{9}(\Lambda^2+m_2^2)^{\frac{3}{2}} + 5m_2^2(\Lambda^2+m_2^2)^{\frac{1}{2}}\right)$$
$$= \left[\frac{1}{9}\frac{22}{3} + \frac{2}{27}\frac{2\cdot 5\cdot 11}{9} + \ldots\frac{2}{9} + \frac{20}{81} + \ldots\right]\Lambda^2$$
$$+ \left[\frac{1}{9}\frac{22}{3}\frac{3}{2}m_2^2 + \frac{5}{9}m_2^2 + \frac{2}{27}\frac{2\cdot 5\cdot 11}{9}\frac{5}{2}m_2^2 - \frac{2}{27}\frac{160}{9}m_2^2 + \frac{2}{9}\frac{1}{2}m_2^2 + \frac{20}{81}\frac{3}{2} + \frac{20}{9}m_2^2 + \ldots\right] + \ldots$$

and the constant in the integral of $\sqrt{\frac{5}{3}\Lambda^2 + 5m_2^2}$. Then

(40)

$$I_1(0)_{finite} = -\frac{2\cdot 5^2 g^2}{3^2\pi^3}\left\{\left[-\frac{\sqrt{15}\pi}{8} + \frac{5}{2}\sqrt{\frac{3}{5}} - \frac{5}{6}\sqrt{\frac{3}{5}} + 2 - \left(\frac{13}{6}\sqrt{\frac{5}{3}} - \frac{1}{2}\right) + \ldots\right] + \ldots \right.$$
$$\left. + \frac{3}{2}\left[\frac{1}{9}\frac{22}{3}\frac{3}{2} + \frac{5}{9} + \frac{2}{27}\frac{2\cdot 5\cdot 11}{9}\frac{5}{2} - \frac{2}{27}\frac{160}{9} + \frac{2}{9}\frac{1}{2} + \frac{20}{81}\frac{3}{2} + \frac{20}{9} + \ldots\right]\right\} m_2^2.$$

The expression of $|\widehat{I_1(0)}|_{finite}$ to the given order is 1.00605044385 $\frac{m_2^2}{\mu_{QCD}^2} g^2$. The coupling in the pseudoscalar meson theory is nearly $g = 0.2$. Setting $g = 0.2 z_g$ and m_2 to be the weighted average of masses of quarks, $\frac{1}{6}m_u + \frac{1}{6}m_d + \frac{2}{3}m_s$, this value equals 0.1833580682 z_g^2. The quantum corrected pseudoscalar mass is

(41)

$$m_{1q} \approx m_1 \left\{1 - 3.0364440741\ sin(0.06038577145\ z_g^2)\right.$$

(42)

$$\left[3.0364440741\ sin(0.06038577145\ z_g^2) + 0.1833580682\ z_g^2\right.$$
$$\left.\cdot\left[1 - 5.85491221095 z_g^{-2}\ \left(sin(0.08539837695 z_g^2) + sinh(0.08539837695 z_g^2)\right)\right]\right]\Bigg\}^{-\frac{1}{2}}.$$

Setting m_1 equal to the η-meson mass $m_\eta = 547.862 \pm 0.018\ MeV/c^2$ and $m_{1q} = m_{\eta'} = 957.78 \pm 0.06\ MeV/c^2$,

(43) $$z_g = 2.127038 \pm 0.000086.$$

This value of z_g yields the coupling

(44) $$g = 0.4254076 \pm 0.0000172.$$

There remains a theoretical uncertainty in the exact value of the series in Eq.(39). This value of g is compatible with the coupling of the η meson to the strange quark within the current margin of error [6].

Several different models have been developed for η' meson. Amongst these are the pion bubble [7] and gluon condensate models [8],[9],[10]. In the first, the mass is identified with that of seven pion bubbles according to the integer multiple rule [11]. The second introduces a nonperturbative correction to the massive quark propagator, and twice of this mass differential which can be added to the value of the mass for $\frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$ from the 0^- spin coupling formula. The sum

$$(45)\quad \frac{1}{3}\left(2m_u-\frac{3A}{4m_u^2}\right)+\frac{1}{3}\left(2m_d-\frac{3A}{4m_d^2}\right)+\frac{1}{3}\left(m_s-\frac{3A}{4m-s^2}\right)+\Delta M(p^2)_{q\bar{q}}$$
$$=349.49476\ MeV/c^2+481.58394\ MeV/c^2=831.08154\ MeV/c^2$$

can be combined with a variant of the octet mass, $\bar{m}_8$, which is the average of the masses derived from the linear Gell-Mann-Okubo formula

(46)
$$\bar{m}_8=\frac{1}{2}\left[\frac{1}{2}\left(\frac{m_{K^-}+m_{\bar{K}_0}}{2}+\frac{m_{K^+}+m_{K^{*0}}}{2}\right)+\frac{3m_\eta+m_\pi}{4}\right]=470.534384167\ MeV/c^2,$$

in quadrature to give a value

(47)
$$\sqrt{(831.08154\ MeV/c^2)^2+(470.534384167\ MeV/c^2)^2}=955.038812202\ MeV/c^2$$

nearly equal to $m_{\eta'}$. The quadratic Gell-Mann-Okubo formula [12] yields

(48)
$$\hat{m}_8=\frac{1}{2}\left\{\frac{1}{2}\left[\left(\frac{m_{K^-}+m_{\bar{K}^0}}{2}\right)^2+\left(\frac{m_{K^+}+m_{K^0}}{2}\right)^2\right]\right\}^{\frac{1}{2}}+\frac{1}{2}\left(\frac{3m_\eta^2+m_\pi^2}{4}\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}(495.6625\ MeV/c^2+479.4567247357\ MeV/c^2)=487.559612368\ MeV/c^2$$

and

(49)

$$\sqrt{(831.08154\ MeV/c^2)^2 + (487.559612368\ MeV/c^2)^2} = 963.540714777\ MeV/c^2,$$

which is larger than $m_{\eta'}$ by $5.76081477697\ MeV/c^2$. Consistency between the two models of the η' meson follows from an annihilation of the quark-anti-quark pairs of the pion bubbles to produce a gluon condensate, which would have an energy that is approximately 40.2% of the 481.58394 MeV energy differential [13], or 193.5967 MeV, equal to 20.213% of the $m_{\eta'}c^2$ [14][15]. This process then produces a fundamental 0^- pseudoscalar core, three quark-anti-quark outer boundary components and gluonic condensate. It may be recalled that the quark boundary components are the ends of surfaces of infinite genus with $m_{u\infty} = 61.85\ MeV/c^2$ and $m_{d\infty} = 64.85\ MeV/c^2$ such the sums with the chiral quark mass of $m_{qc} = 243.15\ MeV/c^2$, resulting from chiral symmetry breaking, are the constituent quark masses $m_u = m_{u\infty} + m_{uc} = 305\ MeV/c^2$ and $m_d = m_{d\infty} + m_{dc} = 308\ MeV/c^2$ [16]. The mass of the quark-anti-quark boundary components then would be

(50)

$$\frac{1}{2}((m_{u\ \infty} + m_{\bar{u}_\infty}) + (m_{d_\infty} + m_{\bar{d}_\infty})) = \frac{1}{2}(2(61.85\ MeV/c^2) + 2(64.85\ MeV/c^2))$$
$$= 126.7\ MeV.c^2.$$

It is known that there are bridges which connect these boundary components. The u and d bridges [17] have masses

(51)

$$m_{ub} = 1.934658\ MeV/c^2$$
$$m_{db} = 4.93509\ MeV/c^2,$$

which coincide with the light quark masses from lattice calculations [18]. With five pairs of both $u-\bar{u}$ and $d-\bar{d}$ bridges, the total mass of the bridges is

(52) $$5m_{ub} + 5m_{db} = 34.34874\ MeV/c^2.$$

The sum of the energies in the gluon condensate model would be

(53)

$$\frac{1}{3}\left(2m_u - \frac{3A}{4m_u^2}\right) + \frac{1}{3}\left(2m_d - \frac{3A}{4m_d^2}\right) + \frac{1}{3}\left(m_s - \frac{3A}{4m - s^2}\right) + \frac{E_{gluon}}{c^2}$$

$$+ 3 \cdot \frac{1}{2}((m_{u\,\infty} + m_{\bar{u}_\infty}) + (m_{d_\infty} + m_{\bar{d}_\infty})) + 5(m_{ub} + m_{db})$$

$$= 349.49476\ MeV/c^2 + 193.5967\ MeV/c^2 + 3(126.7\ MeV/c^2) + 34.34874\ MeV/c^2.$$

$$= 957.539916\ MeV/c^2$$

The method described in this investigation is based similarly on quark-anti-quark loop amplitudes with the coupling reflecting the chirality of the interaction. The pseudoscalar meson theory also introduces a connection between the η and η' meson masses. The decays of the η' meson through $\eta' \to \eta\pi\pi$, which can be predicted by chiral perturbation theory [19],

(54)
$$\eta' \to \eta\pi^0\pi^0 \to \gamma\gamma\pi^0\pi^0 \to 6\gamma$$

$$\eta' \to \eta\pi^0\pi^0 \to 4\pi^0 \to 8\gamma$$

$$\eta' \to \eta\pi^0\pi^0 \to 5\pi^0 \to 10\gamma$$

over $9.02032 \pm 0.87439 \times 10^{-17}$ *sec* and

(55) $\eta' \to \eta\pi^+\pi^- \to \pi^0\pi^0\pi^+\pi^- \to 4\gamma\mu^+\nu_\mu\mu^-\bar{\nu}_\mu \to 4\gamma e^+e^-2\nu_\mu 2\bar{\nu}_\mu\nu_e\bar{\nu}_e$

$\eta' \to \eta\pi^+\pi^- \to \pi^0\pi^0\pi^0\pi^+\pi^- \to 6\gamma\mu^+\nu_\mu\mu^-\bar{\nu} \to 6\gamma e^+e^-2\nu_\mu 2\bar{\nu}_\mu\nu_e\bar{\nu}_e$

$\eta' \to \eta\pi^+\eta^- \to \pi^0 2\pi^+ 2\pi^- \to 2\gamma 2\mu^+ 2\nu_\mu 2\mu^- 2\bar{\nu}_\mu \to 2\gamma 2e^+ 2e^- 4\nu_\mu 4\bar{\nu}_\mu 2\nu_e 2\bar{\nu}_e$

$\eta' \to \eta\pi^+\pi^- \to 2\gamma\mu^+\nu_\mu\mu^-\bar{\nu}_\mu \to 2\gamma e^+e^-2\nu_\mu 2\bar{\nu}_\mu\nu_e\bar{\nu}_e$

over $2.2230011 \pm 0.0025630 \times 10^{-6}$ *sec*, together with other rarer reactions, belong to semiclassical time scales greater than or equal to 2×10^{-19} *sec*. Furthermore, a pion -induced conversion of an η meson into an η' meson before the decay of the η meson after 5×10^{-19} can occur these time scales. This transformation is essential for the mixing of η and η' mesons. The value of the mixing angle θ_P, consistent with dual models of hadrons is -19.5° [20]. Recent experimental values are $-16.7^\circ \pm 1.0^\circ \pm 2.0^\circ$ [21],-$18.76^\circ \pm 3.4^\circ$ [22] and $-17.2^\circ \pm 0.5^\circ$ [23]. It would be expected that duality is not precisely preserved in a semiclassical theory. However, the pseudoscalar mixing angle can be deduced from an effective field theory with both kinetic and mass matrices [24], which is related to the use of two mixing angles [25] and the chiral Lagrangian [26] and compatible with the change in the coefficients of kinetic terms in the semiclassical action.

4 Conclusion

The pseudoscalar meson octet includes the pion, kaon and the η prime meson A ninth 0^- resonance is the η' meson. The quantum theory of pseudoscalar mesons near semiclassical scales is investigated with regard to the η and η' mesons. There are two masses in the theory, which are designated to be m_1 for the pseudo scalar

field and m_2 for the fermion field. Fermion loop diagram is evaluated with propagators that are constructed for the semiclassical model. Traces of gamma matrices including the γ_5 factor are evaluated and a series expansion of the fermion loop amplitude at zero external momentum has terms quadratic and logarithmic in the momentum cut-off Λ. Renormalizability of the theory is established because the superficial degree of divergence of all of the one-particle irreducible diagrams does not exceed 2. The fermion self-energy energy is linearly divergent. The infinities in this one-loop amplitude, rainbow diagrams and other graphs with internal fermion loops may be absorbed through a renormalization of the fermion mass.

The pseudoscalar momentum-space propagator is modified by perturbative corrections. Even though only one-particle irreducible amplitudes are required for renormalizability, reducible diagrams are relevant for computation of mass shifts in momentum-space propagators. Evaluating the finite parts of the two-point pseudoscalar amplitudes, a theoretical prediction can be made for a summation over all orders of quantum corrections to the η meson mass that coincides with the η' meson mass at a specific value of the coupling of η to the quarks.

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